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QUANTUM COHOMOLOGY AND S^1-ACTIONS WITH ISOLATED FIXED POINTS

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Abstract. This paper studies symplectic manifolds that admit semi-free circle actions with isolated fixed points. We prove, using results on the Seidel element, that the (small) quantum cohomology of a $2n$-dimensional manifold of this type is isomorphic to the (small) quantum cohomology of a product of n copies of $\mathbb{P}^1$. This generalizes a result due to Tolman and Weitsman.

1. Introduction

Let (M, ω) be a $2n$-dimensional compact, connected, symplectic manifold, and let $\{\lambda_t\} = \lambda : S^1 \longrightarrow \mathrm{Symp}(M, \omega)$ be a symplectic circle action on M, that is, if X is the vector field generating the action, then $\mathcal{L}_X\omega = d\iota_X\omega = 0$. Recall that the action is semi-free if it is free on $M \backslash M^{S^1}$. This is equivalent to saying that the only non-zero *weights* at every fixed point are ± 1. A circle action is said to be Hamiltonian if there is a C^∞ function $H : M \longrightarrow \mathbb{R}$ such that $\iota_X\omega = -dH$. Such a function is called a Hamiltonian for the action.

Tolman and Weitsman proved in [10] that if the action is semi-free and admits only isolated fixed points, then the action must be Hamiltonian provided that there is at least one fixed point. There is a great deal of information concerning the topology of manifolds carrying such actions. The first result in this direction is due to Hattori [2]. He proves that there is an isomorphism from the cohomology ring $H^*(M; \mathbb{Z})$ to the cohomology ring of a product of n copies of $\mathbb{P}^1$. Moreover, this isomorphism preserves Chern classes. In [10] Tolman and Weitsman generalize Hattori's result to equivariant cohomology. The main result of this paper is to provide an extension to quantum cohomology. In §3.1 we prove that M is an almost Fano manifold, therefore we can use polynomial coefficients $\Lambda := \mathbb{Q}[q_1, \ldots, q_n]$ for the quantum cohomology ring. The main theorem is the following.

Theorem 1.1. *Let (M, ω) be a $2n$-dimensional compact connected symplectic manifold. Assume M admits a semi-free circle action with a finite non-empty set of fixed points. Then there is an isomorphism of (small) quantum cohomology*

$$QH^*(M; \Lambda) \cong QH^*((\mathbb{P}^1)^n; \Lambda).$$

Note that we can directly compute the quantum cohomology of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ to get the following result.

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Corollary 1.2. *The (small) quantum cohomology of M is given by*

$$QH^*(M;\Lambda) \cong QH^*((\mathbb{P}^1)^n;\Lambda) \cong \frac{\mathbb{Q}[x_1,\ldots,x_n,q_1,\ldots,q_n]}{\langle x_i * x_i - q_i\rangle},$$

where $\deg(x_i) = 2$ *and* $\deg q_i = 4$.

Moreover, all other products are given by

$$x_{i_1} * \cdots * x_{i_k} = x_{i_1} \smile \cdots \smile x_{i_k}$$

for $i_1 < \cdots < i_k$. *Here the product on the left is the quantum product, while the term on the right is the usual cup product.*

To prove Theorem 1.1 we will construct a set of generators $\{x_i\}$ of the cohomology ring $H^*(M;\mathbb{Z})$. Then we prove in Lemma 4.1 that the quantum products of these generators satisfy the expected relations given in Corollary 1.2.

The proof of Theorem 1.1 strongly relies on the techniques and results developed by McDuff and Tolman on the Seidel automorphism of the quantum cohomology of symplectic manifolds with circle actions [4]. We apply their results in our particular case and specialize them to understand exactly how the Seidel automorphism acts on the generators $\{x_i\}$. We will see in Theorem 3.13 that this action does not have higher order terms, that is, the automorphism acts by single homogeneous terms in quantum cohomology. Thus the Seidel automorphism is essentially a permutation of the elements in the basis. We then use this and the associativity of the quantum product to compute the quantum products of the basis $\{x_i\}$. The construction of this basis is based on the tools that Tolman and Weitsman developed to prove the following theorem.

Theorem 1.3 ([10])**.** *Let (M,ω) be a compact, connected symplectic manifold with a semi-free, Hamiltonian circle action with isolated fixed points. Let y be the canonical generator of $H^*(BS^1,\mathbb{Z})$. Then, there is an isomorphism of rings $H^*_{S^1}(M) \simeq H^*_{S^1}(((\mathbb{P}^1)^n))$ which takes the equivariant Chern classes of M to those of $(\mathbb{P}^1)^n$. Therefore the equivariant cohomology ring is given by*

$$H^*_{S^1}(M) = \mathbb{Z}[x_1,\ldots,x_n,y]/(x_i y - x_i^2).$$

Here $x_i \in H^2_{S^1}(M)$ and the equivariant Chern series is given by $c_t(M) = \sum_i c_i(M)t^i$ where

$$c_t(M) = \prod_i (1 + t(2x_i - y)).$$

Although Tolman and Weitsman use equivariant cohomology for getting an invariant base for $H^*(M;\mathbb{Z})$, the results of McDuff-Tolman require a more geometric description of the basis. Therefore the crucial element in most of the results of this paper is having geometric representatives of the cycles dual to the cohomology basis. These geometric representatives are defined by the Morse complex of the Hamiltonian function.

The paper is organized as follows. All the Morse theoretical constructions are in §2.1. In §2.2 we use equivariant cohomology to provide an invariant basis for cohomology. Then we establish the relation with the Morse cycles. In §3.1 we define the quantum cohomology ring, and we get results that help to reduce the quantum product formulas. In §3.3 we define the Seidel automorphism in quantum cohomology. In §3.4 we relate the Seidel automorphism with invariant chains, then we explicitly compute the Seidel element. Finally in §3.5 we provide the proof of Theorem 1.1.

2. Morse theory and equivariant cohomology

In this section we establish some of the tools we need to prove Theorem 1.1. We start in §2.1 with basic definitions of Morse theory. For more details the reader can consult [1, 7].

Following the approach of [4], in §2.2 we will construct invariant Morse cycles to be able to calculate the Seidel element of M. This will be done in Section 3.4. We introduce equivariant cohomology to identify a basis in cohomology and describe the relation with Morse cycles. At the end, we provide several results that will be necessary in §3.

2.1. **Morse theory.** As in §1, let (M, ω) be a symplectic $2n$-dimensional manifold with a S^1 action generated by a Hamiltonian function H. Thus $\iota_X \omega = -dH$ and $X = J\text{grad}(H)$, where the gradient is taken with respect to the metric $g_J(x, y) = \omega(x, Jy)$ for an ω-compatible S^1-invariant almost complex structure J. With respect to this metric, H is a (perfect) Morse function [3], and the zeroes of X are exactly the critical points of H. For each fixed point $p \in M^{S^1}$, denote by $\alpha(p)$ the index of p and let $m(p)$ be the sum of weights at p. Since the action is semi-free $m(p) = n_+(p) - n_-(p)$, where $n_+(p)$ is the number of positive weights and $n_-(p)$ the number of negative ones. Then $\alpha(p) = 2n_-(p) = n - m(p)$.

In order to understand the (co)homology of M in terms of S^1-invariant cycles, we will consider the *stable* and *unstable* manifolds with respect to the gradient flow $-\text{grad}(H)$. More precisely, let p, q be critical points of H. Define the stable and unstable manifolds by

$$W^s(q) = \{\gamma : \mathbb{R} \longrightarrow M | \lim_{t \longrightarrow \infty} \gamma(t) = q\},$$
$$W^u(p) = \{\gamma : \mathbb{R} \longrightarrow M | \lim_{t \longrightarrow -\infty} \gamma(t) = p\}.$$

Here $\gamma(t)$ satisfies the negative gradient flow equation

$$\gamma'(t) = -\text{grad}H(\gamma(t)).$$

These spaces are manifolds of dimension

$$\dim W^s(q) = 2n - \alpha(q) \quad \text{and} \quad \dim W^u(p) = \alpha(p),$$

and the evaluation map $\gamma \mapsto \gamma(0)$ induces smooth embeddings into M:

$$E_q : W^s(q) \longrightarrow M \quad \text{and} \quad E_p : W^u(p) \longrightarrow M.$$

When these manifolds intersect transversally for all fixed points p, q, the gradient flow is said to be Morse-Smale [1, 7]. Under this circumstance we say that the pair (H, g_J) is *Morse regular.*

In [7] Schwarz proved that there is a way of *partially compactifying* the stable and unstable manifolds and that there are natural extensions of the evaluation maps so that these compactifications with their evaluation maps $E_p : \overline{W^s(p)} \longrightarrow M$ and $E_q : \overline{W^u(q)} \longrightarrow M$ define *pseudocycles.* The compactification of $W^s(p)$ is made by adding *broken trajectories* through fixed points of index $\alpha(p) - 1$. When the action is semi-free and admits isolated fixed points, all the fixed points have even index, therefore $W^s(p)$ is already compact in the sense of Schwarz. Thus $W^s(p)$ is itself a pseudocycle. The same is true for $W^u(q)$. It is well known that pseudocycles define classes in homology (see [6]). We will denote by $[W^u(q)] \in H_{\alpha(q)}(M; \mathbb{Z})$ and $[W^s(p)] \in H_{n-\alpha(p)}(M; \mathbb{Z})$ the homology classes defined by these manifolds. To get transversal cycles representing these classes, we need to consider a special type of invariant compatible almost complex structures, as we explain below.

Assume (M,ω) admits a Hamiltonian S^1-action with isolated fixed points. Each fixed point $p \in M$ has a neighborhood $U(p)$ that is diffeomorphic to a neighborhood of zero in a $2n$-dimensional Hermitian vector space $E(p) = E_1 \oplus \cdots \oplus E_n$, in such a way that the moment map H is given by

$$H(v_1, \ldots, v_n) = \sum_j \pi m_j |v_j|^2$$

and S^1 acts on E_j just by multiplication by $e^{2\pi i m_j}$. Here the numbers $m_j \in \mathbb{Z}$ are exactly the weights of the action. Under the identification above, the almost-complex structure J is the standard complex structure on the Hermitian vector space $E(p)$. Observe that $E(p)$ can be written as $E^+ \oplus E^-$, where $E^\pm$ is the sum of the E_j, where $m_j > 0$ or $m_j < 0$, respectively. We can call the spaces $E^\pm$ the positive and negative normal bundles to the point p.

If we start with any compatible almost complex structure J_F near the fixed points, we can extend J to an S^1-invariant ω-compatible almost complex structure J_M on M whose restriction to the open sets $U(p)$ is J_F. Denote by $\mathcal{J}_{\text{inv}}(M)$ the set of all J that are equal to J_M near the fixed points.

The following lemma shows that it is possible to acquire regularity with generic almost-complex structures.

Lemma 2.1 ([4])**.** *Suppose that H generates a semi-free S^1-action on (M,ω). Then for a generic choice of $J \in \mathcal{J}_{\text{inv}}(M)$ the pair (H, g_J) is Morse regular.*

For the rest of this paper, we will only consider Morse regular pairs (H, g_J) as in the previous lemma. We finally remark that when M is equipped with a regular pair and if there is a (broken) gradient trajectory from a fixed point p to a fixed point q, then $\alpha(p) - \alpha(q) > 0$.

2.2. **Equivariant cohomology.** We can start with a quick review of equivariant cohomology. Let ES^1 be a contractible space where S^1 acts freely, and denote $BS^1 = ES^1/S^1$. Then $H^*(BS^1;\mathbb{Z})$ is the polynomial ring $\mathbb{Z}[y]$, where $y \in H^2(BS^1;\mathbb{Z})$.

Let S^1 act on a manifold M. The equivariant cohomology of M, denoted by $H^*_{S^1}(M)$, is defined by $H^*(M \times_{S^1} ES^1;\mathbb{Z})$. Note that $H^*(BS^1;\mathbb{Z})$ is naturally isomorphic to $H^*_{S^1}(pt)$ if $pt \in M$ is a point. Under this construction, we have two natural maps, the projection $p : M \times_{S^1} ES^1 \longrightarrow BS^1$ and the inclusion (as fiber) $i : M \longrightarrow M \times_{S^1} ES^1$. The pullback $p^* : H^*(BS^1;\mathbb{Z}) \longrightarrow H^*_{S^1}(M)$ makes $H^*_{S^1}(M)$ a $H^*(BS^1;\mathbb{Z})$ module, while the restriction $i^* : H^*_{S^1}(M) \longrightarrow H^*(M)$ is the "reduction" of invariant data to ordinary data. An immediate consequence is that $i^*(y) = 0$.

Let $j : M^{S^1} \longrightarrow M$ be the natural inclusion. In [3] Kirwan proved that if the action is Hamiltonian, the induced map $j^* : H^*_{S^1}(M) \longrightarrow H^*_{S^1}(M^{S^1})$ is injective. The proof of this theorem is based on the following result, where we weaken the statement to match our needs. For a fixed point $p \in M^{S^1}$ we denote it by $a|_p := (j_p)^*(a)$, where $(j_p)^* : H^*_{S^1}(M) \longrightarrow H^*_{S^1}(p)$ and j_p is the obvious inclusion.

Theorem 2.2 ([3])**.** *Let the circle act on a symplectic manifold M in a Hamiltonian way. Assume the action is semi-free and that there are only isolated fixed points. Let*

$p \in M$ be a fixed point of index $2k$. Then there exists a unique class $a_p \in H^{2k}_{S^1}(M)$ such that $a_p|_p = (-1)^k y^k$, and $a_p|_{p'} = 0$ for all other fixed points p' of index less than or equal to $2k$. Moreover, if we consider all fixed points, the classes a_p form a basis for $H^*_{S^1}(M)$ as a $H^*(BS^1;\mathbb{Z})$ module.

As a remark on the previous theorem, note that the term $(-1)^k y^k$ is the equivariant Euler class of the negative normal bundle at p.

As stated in §1, there is an isomorphism $H_*(M;\mathbb{Z}) \cong H_*(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1;\mathbb{Z})$ if M satisfies the hypothesis of Theorem 2.2. Since H is perfect there are exactly $\dim(H_{2k}(M)) = \binom{n}{k}$ critical points of index $2k$. In [2, 10], the above isomorphism is proved by counting fixed points. We will not discuss the proof here.

Denote the points of index 2 by $p_1, \dots, p_n$. In light of Theorem 2.2, for each fixed point we get classes $a_1, \dots, a_n \in H^2_{S^1}(M)$ such that

$$\begin{aligned} a_j|_{p_j} &= -y, \\ a_j|_p &= 0 \qquad \text{for all other fixed points } p \text{ of index 0 or 1.} \end{aligned} \tag{1}$$

These classes satisfy the following proposition.

Proposition 2.3 ([10, Prop. 4.4])**.** *Let I be a subset of $\{1, \dots, n\}$ with k elements. There exists a unique fixed point p_I of index $2k$ such that*

$$a_j|_{p_I} = -y \quad \textit{if and only if} \quad j \in I$$

and $a_j|_{p_I} = 0$ otherwise.

Proposition 2.3 identifies the fixed points in M with subsets I of $\mathcal{S} := \{1, \dots, n\}$. Observe that the cohomology class $a_I := \prod_{i \in I} a_i \in H^{2k}_{S^1}(M)$ is the same as the class a_{p_I} mentioned in Theorem 2.2. Moreover this class is such that

$$a_I|_{p_J} = (-1)^k y^k \text{ if and only if } I \subseteq J, \tag{2}$$

and it is zero otherwise.

Remark 2.4. The class a_0, associated to the unique point of index zero, takes the value $1 \in H^0_{S^1}(pt)$ when restricted to any fixed point. Therefore it is the identity element in the ring $H^*_{S^1}(M)$. Denote ya_0 by y.

If we apply the same results to the Hamiltonian function $-H$, we obtain unique classes $b_J \in H^{2n-2k}_{S^1}(M)$ associated to each p_J of index $2k$ such that $b_J|_{p_J} = (-1)^{n-k} y^{n-k}$ and is zero when restricted to all other fixed points of index greater than or equal to $2k$. These classes also form a basis of $H^*_{S^1}(M)$. The next proposition establishes the relation with the former basis.

Proposition 2.5. *Let $I = \{i_1, \dots, i_k\}$ and let $I^c = \{i_{k+1}, \dots, i_n\}$ be its complement. Then the classes b_I satisfy the following relation:*

$$b_I = \sigma_{n-k} + y\sigma_{n-k-1} + \cdots + y^{n-k}, \tag{3}$$

where σ_i is the i-th symmetric function in the variables $a_{i_{k+1}}, \dots, a_{i_n}$.

Proof. By Proposition 2.3 the class $a_i + y$ is such that

$$(a_i + y)|_{p_J} = y \text{ if } i \notin J \text{ and zero otherwise.}$$

These are exactly the relations that characterize the class $b_{\{i\}^c}$; then

$$b_{\{i\}^c} = a_i + y.$$

Now, it follows that

$$b_I = \prod_{i \notin I} b_{\{i^c\}} = \prod_{i \notin I} (a_i + y).$$

This proves the result. □

Consider a point p_I of index $2k$ and associate the class $a_I \in H^{2k}_{S^1}(M)$ as before. When we restrict a_I to M we obtain a class $a_I|_M \in H^{2k}(M;\mathbb{Z})$. By taking the Poincaré dual of $a_I|_M$, we get a homology class $p_I^+ \in H_{2n-2k}(M;\mathbb{Z})$. Similarly, using the class b_I we get a homology class $p_I^- \in H_{2k}(M;\mathbb{Z})$. Here is an immediate corollary of Proposition 2.5.

Corollary 2.6. *The class p_I^- is the same as the class $p_{I^c}^+$.*

Proof. This is clear from (3), because the variable y is mapped to zero under restriction to usual cohomology. Now use the fact that $\sigma_{n-k} = a_{I^c}$. □

The last part of this section establishes the relation of the $p_I^{\pm}$ classes with the stable and unstable manifolds of §2.1. This is summarized in the following proposition. Remember that we are working with an almost-complex structure J in $\mathcal{J}_{\text{inv}}(M)$. This result would fail without this hypothesis.

Proposition 2.7. *Let p_I be a fixed point of index $2k$. Then the classes p_I^- and p_I^+ are exactly the same as the classes $[W^u(p_I)]$ and $[W^s(p_I)]$, respectively.*

Proof. Recall that ES^1 can be taken to be the infinite-dimensional sphere S^∞. Consider a finite-dimensional approximation $M^N := M \times_{S^1} S^{2N+1}$ of $M \times_{S^1} ES^1 = M \times_{S^1} S^\infty$ for $N \in \mathbb{N}$ big enough. These are finite-dimensional smooth compact manifolds. Since $W^s(p_I)$ is S^1-invariant, there is a natural extension $W^{N,s}(p_I) := W^s(p_I) \times_{S^1} S^{2N+1}$ of $W^s(p_I)$ to M^N. Let X^N be the Poincaré dual of $W^{N,s}(p_I)$ in M^N.

For all N, there is a natural inclusion (as fibre) $i_N : M \hookrightarrow M^N$. Since the inclusions are natural, the restriction $X^N|_M := (i_N)^*(X^N) \in H^*(M)$ is the same as the Poincaré dual of $[W^s(p_I)]$ in M.

Observe that the natural inclusions

$$M^N \hookrightarrow M^{N+1} \hookrightarrow \cdots \hookrightarrow \lim_N M^N = M \times_{S^1} ES^1$$

induce a sequence

$$\cdots \longrightarrow X^{N+2} \longrightarrow X^{N+1} \longrightarrow X^N$$

given by the restrictions. Thus, by considering the directed limit, there is an element

$$X := \lim_N X^N \in H^*(M \times_{S^1} ES^1) = H^*_{S^1}(M)$$

that restricts to X^N for all N. Naturally, if $i : M \hookrightarrow M \times_{S^1} ES^1$ is the inclusion, then $X|_M := i^*(X) = \text{PD}([W^s(p_I)])$. We claim that X satisfies the same properties as the class a_I, that is, $X|_{p_I} = (-1)^k y^k$ and $X|_p = 0$ for all other fixed points p such that $\alpha(p) \leq 2k$. Therefore, by Theorem 2.2 we must have $X = a_I$. Then $\text{PD}(X|_M) = \text{PD}(a_I|_M)$, and the result will follow immediately.

Take a neighborhood $U(p_I)$ around p_I as in §2.1. Thus, $U(p_I)$ is isomorphic to an open neighborhood V of zero in $E^+ \oplus E^-$. It is clear that if $U(p_I)$ is small enough, $W^s(p_I) \cap U(p_I)$ is diffeomorphic to $E^+ \cap V$. Therefore, the normal bundle of $W^s(p_I)$ can be locally identified with E^-. Finally, by carrying this localization to X^N and considering the limit, we have $X|_{p_I} = e(E^-) = (-1)^k y^k$, where $e(E^-)$ is the equivariant Euler class of E^-.

To finish the proof, observe that if p is any other fixed point with index less than or equal to $2k$, there is no gradient line from p to p_I. This is because the gradient flow is Morse-Smale. Hence, by using the localization again we obtain that $X|_p = 0$. This proves the proposition. □

Corollary 2.8. *By the definition of the classes a_I and b_I, we have*

$$[W^u(p_I)] = p_I^- = \mathrm{PD}(b_I|_M) \text{ and } [W^s(p_I)] = p_I^+ = \mathrm{PD}(a_I|_M),$$

therefore the product $[W^u(p_I)] \cap [W^s(p_J)]$ is given by

$$[W^u(p_I)] \cap [W^s(p_J)] = \mathrm{PD}(b_I|_M) \cap \mathrm{PD}(a_J|_M) = \mathrm{PD}(b_I a_J|_M).$$

Corollary 2.9. *By Corollary* 2.6 *and Proposition* 2.7 *we have the "duality" relation* $[W^u(p_I)] = [W^s(p_{I^c})]$.

Remark 2.10. Let $x_i := a_i|_M = \mathrm{PD}(p_i^+) \in H^2(M;\mathbb{Z})$. The theory of this section proves that the elements x_i generate the algebra $H^*(M;\mathbb{Z})$. Therefore a $\mathbb{Z}$-basis for $H^{2k}(M;\mathbb{Z})$ consists of the elements $x_I = x_{i_1} \cdots x_{i_k}$ for sets $I = \{i_1 < i_2 < \cdots < i_k\}$. Moreover, by Theorem 1.3 the first Chern class of M is given by

$$c_1(M) = 2(x_1 + \cdots + x_n),$$

thus the Chern numbers of the spherical classes p_k^- are $c_1(p_k^-) = 2$ for all $k = 1, \ldots, n$.

Proposition 2.7 also provides some information about the existence of gradient lines. More precisely we have the next proposition.

Proposition 2.11. *Let $I = \{i_1, \ldots, i_k\} \subset \mathcal{S}$. Take $i_{k+1} \notin I$ and consider $I' = I \cup \{i_{k+1}\}$. Let $A_I := \sum_{i \in I} p_i^- \in H_2(M)$. Then:*

a) *There is a gradient line from $p_{I'}$ to p_I. Moreover, the homology class of the sphere generated by rotating the gradient line by the S^1 action is $p_{i_{k+1}}^-$.*
b) *There is a broken gradient line from $p_{\mathcal{S}}$ to p_I. The class A_{I^c} is then represented by rotating this broken line and then $c_1(A_{I^c}) = n + m(p_I)$.*

Proof. To prove there is a gradient line from $p_{I'}$ to p_I, we need to show that the intersection $W^u(p_{I'}) \cap W^s(p_I)$ is non-empty. By definition of the intersection product in terms of pseudocycles [6], it is enough to prove that the intersection product of the classes $[W^u(p_{I'})]$ and $[W^s(p_I)]$ is non-zero.

Consider the equivariant cohomology classes $b_{I'}$ and a_I. By Proposition 2.5 we get

$$b_{I'} a_I = a_{I'^c} a_I + yd,$$

where $d \in H^*_{S^1}(M)$. Since $I'^c \cup I = \{i_{k+1}\}^c$,

$$a_{I'^c} a_I = a_{\{i_{k+1}\}^c}.$$

Once again by Proposition 2.5

$$a_{\{i_{k+1}\}^c}|_M = b_{i_{k+1}}|_M,$$

thus

$$b_{I'} a_I|_M = b_{i_{k+1}}|_M.$$

Now, using Corollary 2.8 we get

$$[W^u(p_{I'})] \cap [W^s(p_I)] = \mathrm{PD}(b_{I'}a_I|_M) = \mathrm{PD}(b_{i_{k+1}}|_M) = p^-_{i_{k+1}} \neq 0. \tag{4}$$

Therefore, there is a gradient line, thus a whole *gradient sphere* A, just by rotating the gradient line. Note that there can be more than one gradient sphere from $p_{I'}$ to p_I. We claim that all these gradient spheres must be homologous.

It is not hard to see from the construction of A that

$$\omega(A) = \int_A \omega = H(p_{I'}) - H(p_I).$$

Therefore if A' is another gradient sphere joining $p_{I'}$ and p_I, $\omega(A) = \omega(A')$. Also observe that if ω' is any S^1-invariant form sufficiently close to ω, then $\omega(A) = \omega'(A)$. Now since the symplectic condition is an open condition, we can perturb ω to obtain a new symplectic form ω' close to ω. By averaging with respect to the group action, we can assume the form ω' to be S^1-invariant. This proves that the classes A' and A have the same symplectic area, that is, $\omega'(A) = \omega'(A')$ for an open set of symplectic forms ω'. Since M is simply connected and there is no torsion, A must be homologous to A'. Finally by (4) this sphere must be in class $p^-_{i_{k+1}}$. Finally, recall that by Remark 2.10 the Chern number of the class $p^-_{i_{k+1}}$ is given by $c_1(p^-_{i_{k+1}}) = m(p_I) - m(p_{I'}) = 2$.

To prove the second part, we can apply the same process for each point in $I^c = \{i_{k+1}, \ldots, i_n\}$. Then getting a sequence of gradient lines,

$$p_{\mathcal{S}} \xrightarrow{\gamma_1} p_{\mathcal{S}-\{i_{n-1}\}} \cdots p_{I\cup\{i_{k+1}\}} \xrightarrow{\gamma_{n-k}} p_I.$$

It is now clear that the chain of gradient spheres obtained by rotating this broken gradient line must be in class $A_{I^c} = A_{i_{k+1}} + \cdots + A_{i_n}$. Note that we could also use a gluing argument as in [8] to prove that there is an honest gradient line from $p_{\mathcal{S}}$ to p_I. Thus

$$c_1(A_{I^c}) = \sum_{l=k+1}^{n} c_1(A_{i_l}) = m(p_I) - m(p_{\mathcal{S}}) = n + m(p_I).$$

□

Corollary 2.12. *Assume x, y are any fixed points in M such that there is a gradient line joining them. Then the Chern class of the J-holomorphic sphere that is obtained by rotating the gradient line by the action has Chern number $|m(x) - m(y)|$.*

Proof. The proof is based on the same computations of Proposition 2.11. Recall that $m(x) = n - \alpha(x)$, where $\alpha(x)$ is the Morse index of x. Since there is a gradient line joining x and y we can assume that they do not have the same index, otherwise it contradicts the fact that the gradient flow is Morse-Smale. Without loss of generality assume $m(x) < m(y)$, and that $x = p_I, y = p_J$. Following the notation of the previous proposition, let A_{I^c} and A_{J^c} be the classes of the spheres obtained by rotating the gradient lines that join $p_{\mathcal{S}}$ with p_I and $p_{\mathcal{S}}$ with p_J, respectively. Thus, if A is the class of the sphere obtained by rotating the gradient line that joins p_I with p_J, then $A + A_{I^c} = A_{J^c}$. Finally, the Chern numbers satisfy $c_1(A) + c_1(A_{I^c}) = c_1(A_{J^c})$, which in turn by Proposition 2.11 gives $c_1(A) = n + m(p_J) - n - m(p_I) = m(x) - m(y)$. □

3. Quantum cohomology and the Seidel automorphism

3.1. **Small quantum cohomology.** In the literature, there are several definitions of quantum cohomology. In this section we make precise the definition of the quantum cohomology we are using, assuming the definition of genus zero Gromov-Witten invariants. We will follow the approach of [6, Chapter 11] entirely.

Let Λ_ω be the usual *Novikov ring* of (M, ω). We recall that Λ_ω is the completion of the group ring of $H_2(M) := H_2(M;\mathbb{Z})/\text{Torsion}$. It consists of all (possibly infinite) formal sums of the form

$$\lambda = \sum_{A \in H_2(M)} \lambda_A e^A,$$

where $\lambda_A \in \mathbb{R}$ and the sum satisfies the finiteness condition

$$\#\{A \in H_2(M) | \lambda_A \neq 0, \omega(A) \leq c\} < \infty$$

for every real number c. By definition, $\deg(e^A) = 2c_1(A)$, where c_1 is the first Chern class of M.

The **(small) quantum cohomology** of M with coefficients in Λ_ω is defined by

$$QH^*(M) := H^*(M) \otimes_{\mathbb{Z}} \Lambda_\omega.$$

As before $H^*(M)$ denotes the ring $H^*(M;\mathbb{Z})$ modulo torsion. We now proceed to define the **quantum product** on $QH^*(M)$. We want the quantum product to be a linear homomorphism of Λ_ω-modules

$$QH^*(M) \otimes_{\Lambda_\omega} QH^*(M) \longrightarrow QH^*(M) : (a,b) \mapsto a * b.$$

Since $QH^*(M)$ is generated by the elements of $H^*(M)$ as a Λ_ω-module, it is enough to describe the multiplication for elements in $H^*(M)$. Let $e_0, e_1, \dots, e_n$ be a basis for $H^*(M)$ (as a $\mathbb{Z}$-module). Assume each element is homogeneous and $e_0 = 1$, the identity for the usual product. Define the integer matrix

$$g_{ij} := \int_M e_i \smile e_j.$$

Here $e_i \smile e_j$ is the usual cup product in cohomology. Let g^{ij} be the inverse matrix. The quantum product of $a, b \in H^*(M)$ is defined by

$$a * b := \sum_{B \in H_2(M)} \sum_{k,j} \mathrm{GW}^M_{B,3}(a, b, e_k) g^{kj} e_j \otimes e^B. \tag{5}$$

The coefficients $\mathrm{GW}^M_{B,3}$ are the usual Gromov-Witten invariants of J-holomorphic curves in class B. The terms in the sum are non-zero only if $\deg(e_k) + \deg(e_j) = \dim M$ and $\deg(a) + \deg(b) + \deg(e_k) = \dim M + 2c_1(B)$. Thus, it is enough to consider classes B such that

$$\deg(a) + \deg(b) - \dim M \leq 2c_1(B) \leq \deg(a) + \deg(b).$$

In the problem at hand, a basis for $H^*(M)$ is given by the elements x_I as in Remark 2.10. Then the integrals

$$g_{IJ} = \int_M x_I \smile x_J$$

all vanish unless the sets I and J are complementary. This is because if I, $J \subset \{1, \dots, n\}$, $x_I \smile x_J = x_{\mathcal{S}}$ if and only if $I^c = J$. Here $x_{\mathcal{S}}$ is the positive generator of $H^{2n}(M;\mathbb{Z})$.

We claim that to compute the quantum product, we only need to consider in (5) classes B such that $c_1(B) \geq 0$. More precisely, we have the proposition.

Proposition 3.1. *Assume (M, ω) is a symplectic manifold with a semi-free S^1-action with only isolated fixed points. Let $B \in H_2(M)$, and let $a, b, c \in H^*(M)$. If $c_1(B) < 0$, then the Gromov-Witten invariant $\mathrm{GW}^M_{B,3}(a, b, c)$ is zero. Moreover, if $c_1(B) = 0$ and some $\mathrm{GW}^M_{B,3} \neq 0$, then $B = 0$. Therefore, the expression for the quantum product (5) can be written as*

$$a * b = a \smile b + \sum_{B \in H_2(M), c_1(B) > 0} a_B \otimes e^B,$$

where the classes a_B have degree $\deg(a_B) = \deg a + \deg b - 2c_1(B)$.

Remark 3.2. Note that since $c_1(B)$ is even, the classes a_B appear in the sum above by "jumps" of four in the degree.

The rest of this section is dedicated to the proof of Proposition 3.1.

To compute the Gromov-Witten invariants $\mathrm{GW}^M_{B,3}(a, b, c)$ one usually constructs a regularization (virtual cycle) $\overline{\mathcal{M}}^{\nu}_{0,3}(M, J, B)$ of the moduli space $\overline{\mathcal{M}}_{0,3}(M, J, B)$. Then one computes the intersection number of the evaluation map

$$ev : \overline{\mathcal{M}}^{\nu}_{0,3}(M, J, B) \longrightarrow M^3$$

with a cycle $\alpha_1 \times \alpha_2 \times \alpha_3$ representing the class $\mathrm{PD}(a) \times \mathrm{PD}(b) \times \mathrm{PD}(c)$. This procedure can be modified in the following way. First, let $\alpha : Z \longrightarrow M^3$ be a pseudocycle that represents the product $\mathrm{PD}(a) \times \mathrm{PD}(b) \times \mathrm{PD}(c)$, then define the *cut-down* moduli space by

$$\overline{\mathcal{M}}_{0,3}(M, J, B; Z) := ev^{-1}(\overline{\alpha(Z)}).$$

Here $ev : \overline{\mathcal{M}}_{0,3}(M, J, B) \longrightarrow M^3$ is the evaluation map and $\overline{\alpha(Z)}$ is the closure in M^3 of the pseudocycle Z [6]. Finally, construct a regularization of the cut-down moduli space. McDuff and Tolman use this approach to calculate the Gromov-Witten invariants. The next result is proved in [4]; it shows exactly how to compute the invariants $\mathrm{GW}^M_{B,3}$ using this procedure. Remember that an S^1 action on M can be extended to an action on J-holomorphic curves just by post-composition. Also, a pseudocycle $\alpha : Z \longrightarrow M^3$ is said to be S^1-invariant if $\alpha(Z)$ is.

Proposition 3.3. *Let (M, ω) be a symplectic manifold. Then, the Gromov-Witten invariant $\mathrm{GW}^M_{B,3}(a, b, c)$ is a sum of contributions, one from each connected component of the moduli space $\overline{\mathcal{M}}_{0,3}(M, J, B; Z)$.*

Assume now that M is equipped with an S^1 action $\{\lambda_t\}$ and that $\alpha : Z \longrightarrow M^3$ and J are S^1-invariant. Then, a connected component of $\overline{\mathcal{M}}_{0,3}(M, J, B; Z)$ makes no contribution to $\mathrm{GW}^M_{B,3}(a, b, c)$ unless it contains an S^1-invariant element.

Proposition 3.3 shows that to compute the Gromov-Witten invariants in the presence of a circle action, one has to compute the invariant elements in the moduli spaces. The following lemma is a modification of McDuff-Tolman [4, Lemma 3.5]. It describes what the non-constant invariant elements in the moduli space $\mathcal{M}_{0,k}(M, J, B)$ are. We include a proof so that Corollary 3.5 is a natural result.

Lemma 3.4. *Let (M, ω) be a symplectic manifold with a Hamiltonian S^1-action. Let J be an invariant almost complex structure compatible with ω, and let g_J be the*

S^1-invariant metric associated to J. Suppose $[u, z_1, \ldots, z_k]$ is a class in the moduli space $\mathcal{M}_{0,k}(M, J, B)$ represented by a J-holomorphic sphere $u : \mathbb{P}^1 \longrightarrow M$, and k marked points $z_i \in \mathbb{P}^1$. Assume $[u, z_1, \ldots, z_k]$ is fixed by the action $\lambda = \{\lambda_\theta\}$. Then, if $\mathrm{im}(u)$ does not lie entirely in M^{S^1}, there are at most two marked points, i.e. $k \leq 2$, and there are two integers p, q, $p \neq 0, q > 0$, a parametrization $u : \mathbb{R} \times S^1 \longrightarrow M$ and a path $\gamma : \mathbb{R} \longrightarrow M$ joining two fixed points $x, y \in M^{S^1}$ so that the marked points are in $u^{-1}\{x, y\}$ and such that

$$\gamma'(s) = \frac{p}{q}\ \mathrm{grad}(H) \ \textit{and} \ u(s,t) = \lambda_{\frac{pt}{q}}\gamma(s). \tag{6}$$

Moreover, if we fix γ, the parametrization is unique provided

$$x = \lim_{s \longrightarrow -\infty} \gamma(s) \ \textit{and} \ y = \lim_{s \longrightarrow \infty} \gamma(s).$$

Finally, if the action is semi-free, then q is 1.

Proof. If the image of u is in M^{S^1}, then the map is constant. Assume that $u : \mathbb{P}^1 \longrightarrow M$ is a non-constant and not multiply covered J-holomorphic sphere in M. Since the equivalence class $[u, z_1, \ldots, z_k]$ is fixed under the action, for each $\theta \in S^1$ the map $(u', z_1, \ldots, z_k) := (\lambda_\theta \circ u, z_1, \ldots, z_k)$ must be a reparametrization of $(u, z_1, \ldots, z_k)$. Thus, there is a $\phi_\theta \in \mathrm{PSL}(2, \mathbb{C})$ such that $\lambda_\theta \circ u = u \circ \phi_\theta$, and $z_i = \phi_\theta(z_i)$. Therefore $\lambda_\theta(u(z_i)) = u(z_i)$ for all $\theta \in S^1$; then $u(z_i) \in M^{S^1}$ for all i. Since the map u is not multiply covered, ϕ_θ is unique. Then, it is easy to see that the assignment $S^1 \longrightarrow \mathrm{PSL}(2, \mathbb{C}) : \ \theta \mapsto \phi_\theta$ is a homomorphism. Using the fact that the only circle subgroups of $\mathrm{PSL}(2, \mathbb{C})$ are rotations about a fixed axis, we can see that there are exactly two points in $\mathbb{P}^1$ that are mapped by u into M^{S^1}. We can choose coordinates on $\mathbb{P}^1$ so that the rotation axis is the line joining the points $[0 : 1]$ and $[1 : 0]$. As we saw before, the image of any marked point is fixed by the action. Then we have that all the marked points are contained in the set $\{[0 : 1], [1 : 0]\}$. It follows that $k \leq 2$. In the case $\mathrm{Im}(u) \cap M^{S^1} = \{x, y\}$, we may choose $u([0 : 1]) = x$ and $u([1 : 0]) = y$. Identify $\mathbb{P}^1 \backslash \{[0 : 1], [1 : 0]\}$ with the cylinder $\mathbb{R} \times S^1$ with standard coordinates (s, t) and complex structure j_0 defined by $j_0(\partial_s) = \partial_t$, $(s, t) \in \mathbb{R} \times S^1$. If $k = 2$ we identify the marked points $[0 : 1], [1 : 0]$ with the ends of the cylinder. Thus we find that for these coordinates there is a $q \neq 0$ such that

$$\phi_\theta(s, t) = (s, t + q\theta) \text{ and } (\lambda_\theta \circ u)(s, t) = u(s, t + q\theta).$$

Therefore, the isotropy group at any point in $\mathrm{im}(u)$ is given by $\mathbb{Z}_{|q|}$. The sign of q is uniquely determined by the choices we made.

Define $\gamma(s) := u(s, 0)$. Then we get $u(s, t) = \lambda_{t/q}\gamma(s)$. Since u is J-homomorphic and J is invariant,

$$(\lambda_{\frac{t}{q}})_*(\gamma'(s) + \frac{1}{q}JX(\gamma(s))) = \partial_s u + J\partial_t u = 0.$$

With respect to the metric g_J, the gradient flow of H is given by $\mathrm{grad}H = -JX$, thus $\gamma'(s) = \frac{1}{q}\mathrm{grad}(H)(\gamma(s))$. Now use the fact that any sphere is a $|p|$-fold cover of a simple one. We absorb any negative sign into p rather than q.

If the action is semi-free, the isotropy groups are trivial, thus we must have $q = 1$. $\square$

To exemplify the choice of signs in the previous lemma, take $M = \mathbb{P}^1$ with the standard semi-free circle action that rotates M with speed one. Clearly the Hamiltonian is the height function, and the only fixed points are $S := [0 : 1], N := [1 : 0]$. Assume the holomorphic map $u : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ is simple and has two marked points, which as before are just identified with S, N. If we want to parametrize this map, we have to choose a gradient line that joins N and S. Say we have chosen the gradient line that goes from the north pole N to the south pole S in this order (downwards). If we identify $\mathbb{P}^1 \backslash \{S, N\}$ with the cylinder $\mathbb{R} \times S^1$, in the standard coordinates (s, t) the action must rotate positively. That is, we have that q must be 1 and $p = -1$.

Although the unicity of the parametrization is not needed for the proof of the following result, it is good to have a canonical choice of the parametrization. The next corollary will be needed in the proof of Proposition 3.1.

Corollary 3.5. *Assume the same hypothesis as in Lemma* 3.4*, and that the action is semi-free. Let u be an S^1-invariant sphere, and let $A \in H_2(M)$ be its homology class in M. Then, with the parametrization provided by* (6)*, the first Chern number $c_1(A)$ is given by $c_1(A) = |p(m(y) - m(x))|$ with p an integer. Therefore $c_1(A)$ is non-negative.*

Proof. If u is constant, $c_1(A) = 0$. If u is not constant, it must have a parametrization as the one given in Lemma 3.4. That is, for some fixed points x, y in M, and a gradient line γ joining them, u can be parametrized as (6). Assume without lost of generality that $m(x) < m(y)$; then we may choose the path γ from x to y. Then $q = 1$ and p is negative (note that we have chosen the *negative* gradient to be Morse-Smale). This proves that u is a $|p|$-cover of the simple gradient sphere B obtained by rotating the gradient line γ joining x, y. By Corollary 2.12 $c_1(B) = m(y) - m(x)$, and then $c_1(A) = |p|(m(y) - m(x))$. □

Remark 3.6. Let u be an S^1-invariant holomorphic sphere, and let $A \in H_2(M)$ be its homology class. Lemma 3.4 and Corollary 3.5 imply that if $c_1(A) = 0$, then A must be zero. This is because if A joins two fixed points $x, y \in M$, they must have the same index, which is not possible because the flow is assumed to be Morse-Smale.

Note that our original goal was to understand the invariant stable maps in $\overline{\mathcal{M}}_{0,3}(M, J, B; Z)$. By Lemma 3.4, the non-constant components of the stable maps may carry at most two special points. Then the S^1-invariant elements in $\overline{\mathcal{M}}_{0,3}(M, J, B; Z)$ may have a *ghost* component that carries the third marked point.

Proof of Proposition 3.1. By Proposition 3.3 a component of $\overline{\mathcal{M}}_{0,3}(M, J, B; Z)$ contributes to $\mathrm{GW}^M_{B,3}(a, b, c)$ only if the moduli space has an S^1-invariant stable map $\mathbf{u}$. We can assume that for a representative $\{u_i\}$ of $\mathbf{u}$ there is at least one non-trivial component u_α. Since $\mathbf{u}$ is invariant, we can choose the representative $\{u_i\}$ so that each component u_i is invariant (up to reparametrization) under the circle action. This is clear if for instance the representative $\{u_i\}$ does not have any automorphisms that interchange its components. In the case such automorphisms exist, we can always choose a representative such that it is fixed by the action. To exemplify this, assume for simplicity that $\mathbf{u}$ has three components $u_i : \Sigma_i \simeq \mathbb{P}^1 \longrightarrow M, i = 1, 2, 3$, where u_3 is constant and carries a marked point (so that the map is stable), and u_1, u_2 map Σ_1, Σ_2 into the same image in M. Therefore, there is an automorphism of this stable map that permutes the domains Σ_1, Σ_2. Let $\theta \in S^1$. Since λ_θ fixes $\mathbf{u}$, then, by the mere definition of a stable map (Definition 5.1.4 in [6]), we have that $\lambda_\theta \circ u_1 = u_{f_\theta(1)} \circ \psi_\theta$, where ψ_θ is an element in $\mathrm{PSL}(2,\mathbb{C})$ and f_θ is a bijection on the set $\{1,2\}$. Note that f_θ does not need to depend continuously on θ, but in any case we may change the representative of $\mathbf{u}$ so that for all θ, f_θ is the identity. Thus, the components of $\mathbf{u}$ are invariant (up to reparametrization) with respect to the circle action.

Therefore, following the same idea as in the proof of Corollary 3.5 we have that $c_1(B_\alpha) > 0$ if $B_\alpha \in H_2(M)$ is the class that u_α represents. Then $c_1(B) > 0$, and the first claim follows. Note that the second part is a direct consequence of Remark 3.6, because any S^1-invariant J-holomorphic map with zero Chern class must be constant.

Finally, the product $a * b$ can be written as

$$a \smile b + \sum_{c_1(B)>0} \sum_I \mathrm{GW}^M_{B,3}(a,b,x_I)x_{I^c} \otimes e^B.$$

Now take

$$a_B := \sum_I \mathrm{GW}^M_{B,3}(a,b,x_I)x_{I^c}.$$

This proves the proposition. Note that $\deg(a_B) = \deg a + \deg b - 2c_1(B)$. □

3.2. **Almost Fano manifolds.** Assume the hypotheses of Proposition 3.1. The relevant spheres (the ones that count for the GW invariants) all have positive first Chern class. Moreover, let $B \in H_2(M)$ be a class such that some invariant $\mathrm{GW}^M_{B,3} \neq 0$; then $c_1(B) \geq 0$. By Proposition 2.11 and Lemma 3.4, B can be written as a combination

$$B = \sum_i d_i p_i^-,$$

where the coefficients d_i are non-negative integers. Therefore, if we define $A_i := p_i^-$ and $q_i := e^{A_i}$, we may now consider the polynomial ring

$$\Lambda = \mathbb{Q}[q_1, \ldots, q_n]$$

as coefficients for the quantum cohomology. Then, if B is as before,

$$e^B = {q_1}^{d_1} \cdots {q_n}^{d_n}.$$

This will be really useful in §3.5. For the rest of this paper, we will assume Λ to be the quantum coefficient ring.

We finish this section with a discussion about the behavior of J-holomorphic curves in M. In the literature an almost complex manifold (N, J) is said to be **Fano** if the first Chern class $c_1(TN, J)$ takes positive values on the **effective cone** $K^{\mathrm{eff}}(N, J)$, namely

$$K^{\mathrm{eff}}(N,J) := \{A \in H_2(N) | \exists \text{ a } J\text{-holomorphic curve in class } A\}.$$

In symplectic geometry sometimes it is useful to consider the definition

$$K^{\mathrm{eff}}(N,\omega) = \{A \in H_2(N) | \exists A_1, \ldots, A_n \in H_2(N) : A = \sum_i A_i, \mathrm{GW}^M_{A_i,3} \neq 0\}$$

for the effective cone on a symplectic manifold (N, ω, J) with a compatible almost complex structure J. Its clear that $K^{\mathrm{eff}}(N,\omega) \subset K^{\mathrm{eff}}(N,J)$. Then, we can say that

(N,ω,J) is **almost Fano** if the first Chern class $c_1(TN,J)$ takes positive values on the effective cone $K^{\text{eff}}(N,\omega)$. We have the following corollary.

Corollary 3.7. *Let (M,ω) be a symplectic manifold with a semi-free S^1-action with isolated fixed points. Then (M,ω,J) is almost Fano.*

3.3. **The Seidel automorphism.** In this paragraph we introduce the theory behind the definition of the Seidel element. The results concerning the present problem are discussed next. We will follow closely the book [6]. The proofs of the results exposed in this section are mostly contained in Chapters 8, 9 and 11.

Let M be a symplectic manifold with a Hamiltonian circle action. We associate to M the locally trivial bundle $\widetilde{M}_\lambda$ over $\mathbb{P}^1$ with fibre M defined by the *clutching function* (action) $\lambda : S^1 \longrightarrow \text{Ham}(M,\omega)$:

$$\widetilde{M}_\lambda := S^3 \times_{S^1} M.$$

We denote the fibres at $[0:1]$ and $[1:0]$ by M_0 and M_∞, respectively. Note that the isomorphism type of $\widetilde{M}_\lambda$ depends only on the homotopy class of λ.

Since λ is Hamiltonian, we can construct a symplectic form Ω on $\widetilde{M}_\lambda$. In fact the bundle $\pi : \widetilde{M}_\lambda \longrightarrow \mathbb{P}^1$ is a *Hamiltonian fibration* with fibre M, thus admitting sections ([6, Chapter 8]). We choose an Ω-compatible almost complex structure $\widetilde{J}$ on $\widetilde{M}_\lambda$, such that $\widetilde{J}$ is the product $J_0 \times J$ under trivializations. We can define for each fixed point $x \in M^{S^1}$ a $\widetilde{J}$ pseudoholomorphic section $\sigma_x := \{[z_0:z_1;x] | [z_0:z_1] \in \mathbb{P}^1\}$.

$\widetilde{M}_\lambda$ has a canonical cohomology class, the first Chern class of the *vertical tangent bundle* or vertical class $c_{\text{vert}} = c_1(T\widetilde{M}_\lambda^{\text{vert}}) \in H^2(\widetilde{M}_\lambda,\mathbb{Z})$. If x is a fixed point for the circle action, we have that $c_{\text{vert}}(\sigma_x) = m(x)$. This follows from the fact that the normal bundle of the section σ_x is a sum of line bundles $L_i \longrightarrow \mathbb{P}^1$, one for each weight m_i of x and with Chern class $c_1(L_i) = m_i$ (see [4, Lemma 2.2]). Note that if B is a spherical class in M and if M is embedded in $\widetilde{M}_\lambda$ as a fibre, then $c_{\text{vert}}(i_*(B)) = c_1(B)$, where the latter is the usual Chern class of B. Then, for a given fixed point x and its associated section σ_x, the class $\sigma_x + i_*(B)$ has vertical Chern number $m(x) + c_1(B)$.

Take $\widetilde{A} \in H_2(\widetilde{M}_\lambda,\mathbb{Z})$ as a section class, that is, $\pi_*(\widetilde{A}) = [\mathbb{P}^1]$. Let $a_1, a_2 \in H^*(M)$. Given two fixed marked points $\mathbf{w} = (w_1,w_2), w_i \in \mathbb{P}^1$, we may think of the Poincaré dual to the class a_i as represented by a cycle Z_i in the fibre $M_i \hookrightarrow \widetilde{M}_\lambda$ over w_i. With this information it is possible to construct the Gromov-Witten invariant $\text{GW}^{\widetilde{M}_\lambda,\mathbf{w}}_{\widetilde{A},2}(a_1,a_2)$. This invariant counts the number of $\widetilde{J}$-holomorphic sections of $\widetilde{M}_\lambda$ in class $\widetilde{A}$ that pass through the cycles Z_i. This invariant is zero unless $2n + 2c_{\text{vert}}(\widetilde{A}) = \deg(a_1) + \deg(a_2)$. Now we have the following definition.

Definition 3.8. Let (M,ω) be as before. Let $\sigma : \mathbb{P}^1 \longrightarrow \widetilde{M}_\lambda$ be a section, and suppose that σ has vertical class $c = c_{\text{vert}}(\sigma)$. The **Seidel automorphism**

$$\Psi(\lambda,\sigma) : QH^*(M;\Lambda) \longrightarrow QH^{*-2c}(M;\Lambda)$$

is defined by

$$\Psi(\lambda,\sigma)(a) = \sum_{A\in H_2(M)} \sum_{k,j} \text{GW}^{\widetilde{M}_\lambda,\mathbf{w}}_{[\sigma]+i_*(A),2}(a,e_k) g^{kj} e_j \otimes e^A, \tag{7}$$

where $i : M \longrightarrow \widetilde{M}_\lambda$ is an embedding (as fibre).

In this definition we are considering a basis $\{e_i\}$ for $H^*(M)$ as in (5). It is easy to see that the Seidel automorphism as defined above shifts the degree by $-2c$. We just need to analyze when the coefficients in (7) are non-zero. In the particular case of $\sigma = \sigma_x$ for x a fixed point, $\Psi(\lambda, \sigma_x)$ shifts degree by $-2m(x)$. Note that this shift might be positive or negative.

From the previous definition, one can see that an important ingredient for the study of the Seidel element is the moduli space

$$\overline{\mathcal{M}}_{0,2}(\widetilde{M}_\lambda, \tilde{J}, \sigma + i_*(A); Z, Z').$$

This moduli space, as before, is the *cut-off* moduli space of $\tilde{J}$-holomorphic sections in the class $\sigma + i_*(A)$ that pass through cycles Z, Z'. We will say more about these spaces in the next section.

If $\mathbb{1} \in QH^*(M)$ denotes the identity in the quantum cohomology ring, the homogeneous class $\Psi(\lambda, \sigma)(\mathbb{1}) \in QH^*(M)$ is called the **Seidel element** of the action with respect to the section σ. We will use the same notation for the Seidel automorphism and the Seidel element. Thus, the Seidel automorphism is now given just by quantum multiplication by the element $\Psi(\lambda, \sigma)$ [6]. That is,

$$\Psi(\lambda, \sigma)(a) = \Psi(\lambda, \sigma) * a.$$

Note that the Seidel element has degree $\deg(\Psi(\lambda, \sigma)) = 2c_{\text{vert}}(\sigma)$.

3.4. **Seidel automorphism and isolated fixed points.** Now consider the present problem. That is, assume that the action is semi-free and has isolated fixed points. Let $\sigma_{\max}$ be the section defined by the fixed point $p_{\mathcal{S}}$. In this particular case the automorphism $\Psi(\lambda, \sigma_{\max})$ increases the degree by $-2m(p_{\mathcal{S}}) = 2n$. Let $p_I \in M$ be a fixed point. Recall that we can associate to p_I classes in homology p_I^- and p_I^+, and if we consider all the fixed points, then the classes p_I^+ form a basis for $H_*(M)$.

The next theorem, due to McDuff and Tolman [4, Theorem 1.15, Proposition 3.4], gives the first step towards a description of the Seidel automorphism. Although they have proved this result in great generality (the fixed points are allowed to be in submanifolds rather than being isolated) and they use quantum homology rather than cohomology, it is not hard to adapt their result to our present notation.

Theorem 3.9 (McDuff-Tolman)**.** *Let (M, ω) be a symplectic manifold with a semi-free circle action with isolated fixed points. Assume its associated Hamiltonian function H is such that $\int_M H\omega^n = 0$. Let $A_I \in H_2(M)$ be as considered in Proposition* 2.11*. Then, the Seidel automorphism can be expressed as*

$$\Psi(\lambda, \sigma_{\max})(\mathrm{PD}(p_I^-)) = \mathrm{PD}(p_I^+) \otimes e^{A_{I^c}} + \sum_{\omega(B)>0} a_B \otimes e^{A_{I^c}+B},$$

where $a_B \in H^*(M)$. If $a_B \neq 0$, then $\deg \mathrm{PD}(p_I^+) - \deg a_B = 2c_1(B)$. Moreover, if we write the sum above in terms of the basis $\{\mathrm{PD}(p_J^+)\}$, we get

$$\Psi(\lambda, \sigma_{\max})(\mathrm{PD}(p_I^-)) = \mathrm{PD}(p_I^+) \otimes e^{A_{I^c}} + \sum_{\omega(B)>0} \sum_{J \subset \mathcal{S}} C_{B,J}\ \mathrm{PD}(p_J^+) \otimes e^{A_{I^c}+B}.$$

The rational coefficients $C_{B,J}$ can be non-zero only if $|I| - |J| = c_1(B)$ and the moduli space $\overline{\mathcal{M}}_{0,2}(\widetilde{M}_\lambda, \tilde{J}, \sigma_I + i_*(B); \overline{W^u(p_I)}, \overline{W^u(p_J)})$ has an S^1-invariant element. σ_I denotes the section defined by the fixed point p_I.

We know by Corollary 2.6 that $p_I^- = p_{I^c}^+$. By definition $\mathrm{PD}(p_J^+) = x_J$, therefore we have the following straightforward corollary.

Corollary 3.10. *Let (M, ω) be a symplectic manifold with a semi-free circle action with isolated fixed points. Assume its associated Hamiltonian function H is such that $\int_M H\omega^n = 0$. Let $\{x_I\}$ be the basis for the cohomology ring as considered in Remark 2.10, and let $A_I \in H_2(M)$ as considered in Proposition 2.11. The Seidel automorphism can be expressed as*

$$\Psi(\lambda, \sigma_{\max})(x_{I^c}) = x_I \otimes e^{A_{I^c}} + \sum_{\omega(B)>0} \sum_{J \subset \mathcal{S}} C_{B,J}\ x_J \otimes e^{A_{I^c}+B}. \tag{8}$$

The rational coefficients $C_{B,J}$ can be non-zero only if $|I| - |J| = c_1(B)$ and the moduli space $\overline{\mathcal{M}}_{0,2}(\widetilde{M}_\lambda, \tilde{J}, \sigma_I + i_*(B); \overline{W^u(p_I)}, \overline{W^u(p_J)})$ has an S^1-invariant element.

Thus, the key to understanding the Seidel automorphism is first to know what the S^1-invariant elements in moduli spaces

$$\overline{\mathcal{M}}_{0,2}(\widetilde{M}_\lambda, \tilde{J}, \sigma_I + i_*(B); Z, Z')$$

are. Here Z and Z' are closed S^1-invariant cycles in M. These elements are called *invariant chains in section class $\sigma_z + i_*(A)$ from $x \in Z$ to $y \in Z'$ with root z* [4]. We will explain what the meaning of this is. Recall that M is embedded in $\widetilde{M}_\lambda$ as the fibres M_0, M_∞.

Given $x, y, z \in M^{S^1}$, an invariant *principal* chain in section class $\sigma_z + i_*(A)$ from $x \in Z$ to $y \in Z'$ with root z consists of the following:

a) Two sequences of fixed points $\{x = x_1, \ldots, x_k = z\}$, $\{z = y_1, \ldots, y_s = y\}$, where we think of the sequence $\{x_i\}$ as embedded in M_0 and the sequence $\{y_i\}$ in M_∞.
b) The points x_k and y_1 are joined by the section σ_z.
c) For each $1 \le i < k$ and each $1 \le j < s$, the points x_i, x_{i+1} and y_j, y_{j+1} are joined by invariant $\widetilde{J}$-holomorphic spheres in classes $i_*(A_i')$ and $i_*(A_j'')$, respectively. Here the classes A_i', A_j'' are in M.
d) If $A' := \sum_i A_i'$, $A'' := \sum_j A_j''$, then $A = A' + A''$.

An **invariant chain** in section class $\sigma_z + i_*(A)$ from $x \in Z$ to $y \in Z'$ with root z is a chain as above with additional ghost components at each of which a tree of invariant spheres is attached. In this case, A is the sum of classes represented by the principal spheres and the bubbles. As a final remark of this definition, note that since A' is invariant, then $c_1(A') \ge 0$ and $c_1(A') = 0$ if and only if $A' = 0$. The same applies for A''.

An immediate lemma is the following.

Lemma 3.11. *Assume the hypotheses of Corollary 3.10, and suppose $\sigma_z + A$ is an invariant chain in the moduli space*

$$\overline{\mathcal{M}}_{0,2}(\widetilde{M}_\lambda, \tilde{J}, \sigma_I + i_*(B); \overline{W^u(p_I)}, \overline{W^u(p_J)}).$$

As before, $\omega(B) > 0$. Let $A = A' + A''$ be the decomposition of A as described above. Then, the first Chern classes $c_1(A'), c_1(A'')$ can be estimated by

$$c_1(A') \ge |m(x) - m(z)| \text{ and } c_1(A'') \ge |m(y) - m(z)|.$$

Therefore

$$c_1(A) \geq |m(x) - m(z)| + |m(y) - m(z)| \text{ and} \\ c_1(B) \geq c_1(A''). \tag{9}$$

Moreover if the coefficient $C_{B,J} \neq 0$*, then* $c_1(B) > 0$.

Proof. If A_i is an invariant sphere joining x_i to x_{i+1}, Corollary 3.5 shows that for some integers p_i, $c_1(A_i) = |p_i(m(x_i) - m(x_{i+1}))| \geq |m(x_i) - m(x_{i+1})|$. Then $c_1(A') \geq \sum_{i=1}^{k} |m(x_i) - m(x_{i+1})| \geq |m(x) - m(z)|$. The inequality for $c_1(A'')$ follows similarly.

Now, by assumption the invariant chain $\sigma_z + i_*(A)$ is in class $\sigma_I + i_*(B)$, thus $\sigma_z + i_*(A) = \sigma_I + i_*(B)$ and then

$$c_1(B) + m(p_I) = c_1(A) + m(z).$$

Since $x \in \overline{W^u(p_I)}$, $m(x) \geq m(p_I)$. Using $c_1(A) = c_1(A') + c_1(A'')$ and the fact that $c_1(A') \geq m(x) - m(z)$, we get

$$c_1(B) \geq c_1(A'') + m(x) - m(z) - m(p_I) + m(z) \geq c_1(A'') \geq 0.$$

To prove the last claim, we want to see that if $C_{B,J} \neq 0$, then $c_1(B) > 0$. By contradiction assume that $C_{B,J} \neq 0$ and that $c_1(B) = 0$. By Corollary 3.10 there must be an invariant chain $\sigma_z + i_*(A)$ in class $\sigma_I + i_*(B)$, and then $c_1(B) = |I| - |J| = 0$. By (9) we have that $c_1(A'') = 0$. Since A'' is a gradient sphere, $A'' = 0$. As explained before, A'' joins y, z; it follows that $y = z$ and then $m(y) = m(z)$.

Now, from the equality $\sigma_z + i_*(A) = \sigma_I + i_*(B)$ we get $m(z) + c_1(A') - m(p_I) = 0$. Finally, since $y \in \overline{W^u(p_J)}$, $m(z) = m(y) \geq m(p_J) = m(p_I)$, and then

$$0 = c_1(A') + m(z) - m(p_I) \geq c_1(A').$$

Then $A' = 0$, and thus $A = 0$. If $A = 0$, this implies that $x = y = z$. Then $m(x) = m(z) = m(y) = m(p_I)$ and $\sigma_x = \sigma_z + i_*A = i_*B + \sigma_I$. On the other hand, note that if $x = p_I$, then we would have that $\sigma_x = \sigma_I$, thus $i_*B = 0$, which contradicts the assumption $\omega(B) > 0$. Therefore $x \neq p_I$. Since $x \in \overline{W^u(p_I)}$ we have that $m(x) > m(p_I) = m(p_J)$, which is a contradiction. □

With Lemma 3.11 we can simplify the expression (8) to get the following corollary.

Corollary 3.12. *Assume the same hypotheses of Corollary* 3.10*. Then the Seidel element is given by*

$$\Psi(\lambda, \sigma_{\max})(x_{I^c}) = x_I \otimes e^{A_{I^c}} + \sum_{\omega(B)>0, c_1(B)>0} \sum_{J \subset \mathcal{S}} C_{B,J}\ x_J \otimes e^{A_{I^c}+B}. \tag{10}$$

Again $C_{B,J} = 0$ *unless* $|I| - |J| = c_1(B)$*, and the moduli space*

$$\overline{\mathcal{M}}_{0,2}(\widetilde{M}_\lambda, \tilde{J}, \sigma_I + i_*(B); \overline{W^u(p_I)}, \overline{W^u(p_J)})$$

has an S^1*-invariant element.*

Note that the only difference to (8) is that we are considering only classes B with positive Chern number.

If there are any higher order terms, that is, terms that correspond to positive first Chern classes $c_1(B) > 0$, they contribute to the sum (10) as an element of degree $2(|J| + c_1(A_{I^c} + B))$. Heuristically an invariant chain $A + \sigma_z$ makes a contribution only if $c_1(A)$ is big enough so that the inequalities (9) are satisfied. We will see in our next result that with our present hypotheses there are no such contributions. Thus there are no higher order terms. This result fails if for instance we allow the action to have fixed points along submanifolds, as we will see in the example described in §3.5. Observe that we can normalize our Hamiltonian function H (by adding a constant) so that $\int_M H\omega^n = 0$ without altering any of our previous results.

Theorem 3.13. *Let (M,ω) be a symplectic manifold with a semi-free circle action with isolated fixed points. Assume its associated Hamiltonian function H is such that $\int_M H\omega^n = 0$. Then, the Seidel automorphism $\Psi(\lambda, \sigma_{\max})$ acts on the basis $\{x_I\}$ by*

$$\Psi(\lambda, \sigma_{\max})(x_I) = x_{I^c} \otimes e^{A_I}. \tag{11}$$

Proof. Consider I^c instead of I. By Corollary 3.12 the Seidel automorphism can be computed as

$$\Psi(\lambda, \sigma_{\max})(x_{I^c}) = x_I \otimes e^{A_{I^c}} + \sum_{c_1(B)>0, J\subset\mathcal{S}} C_{B,J} x_J \otimes e^{A_{I^c}+B}.$$

As in Proposition 3.1, the Chern number $c_1(B)$ is a multiple of two. Thus the terms in the sum appear with "jumps" of four in the degree. By Corollary 3.12, $C_{B,J}$ is non-zero only if there is an S^1-invariant element in the moduli space $\overline{\mathcal{M}}_{0,2}(\widetilde{M}_\lambda, \tilde{J}, \sigma_I + i_*(B); \overline{W^u(p_I)}, \overline{W^u(p_J)})$. We want to see that the coefficients $C_{B,J}$ are all zero.

By contradiction assume there is an invariant chain $\sigma_z + A$ in this moduli space. Therefore A goes from a fixed point $x \in \overline{W^u(p_I)}$ to a fixed point $y \in \overline{W^u(p_J)}$. This chain satisfies

$$\sigma_z + A = \sigma_I + B. \tag{12}$$

Since the gradient flow is Morse-Smale and there is a gradient line from p_I to x, $m(x) \geq m(p_I) = n - 2|I|$. Analogously $m(y) \geq m(p_J) = n - 2|J|$. Since $c_1(B) = |I| - |J| > 0$ and we know $c_1(A) + m(z) = m(p_I) + c_1(B)$ from (12), we get

$$c_1(A) = 2|K| - |I| - |J|, \tag{13}$$

where $K \subset \mathcal{S}$ is such that $p_K = z$.

Finally, from Lemma 3.11 we have

$$\begin{aligned} c_1(A) &\geq |m(x) - m(z)| + |m(y) - m(z)| \\ &\geq -2m(z) + m(y) + m(x) \\ &\geq 4|K| - 2|I| - 2|J|. \end{aligned}$$

Therefore, by (13)

$$2|K| - |I| - |J| = c_1(A) \geq 2(2|K| - |I| - |J|).$$

This is only possible if $c_1(A) = 0$, i.e. $2|K| - |J| = |I|$. Then, A must be zero and $x = y = z$. Therefore $B = \sigma_z - \sigma_I$ and hence $c_1(B) = m(z) - m(p_I) = 2(|I| - |K|)$. Since $c_1(A) = 0$, (13) implies $|I| - |K| = |K| - |J|$. Thus $0 < c_1(B) = 2(|K| - |J|)$. By hypothesis $p_K = z = y \in \overline{W^u(p_J)}$. Then we have $|K| \leq |J|$. Thus $c_1(B) \leq 0$, which is a contradiction. This proves the theorem. □

Corollary 3.14. *The Seidel element* $\Psi(\lambda, \sigma_{\max})$ *is given by*

$$\Psi(\lambda, \sigma_{\max}) = x_{\mathcal{S}},$$

and the quantum product of $x_{\mathcal{S}}$ *with the element* x_I *is given by*

$$x_{\mathcal{S}} * x_I = x_{I^c} \otimes e^{A_I}. \tag{14}$$

Proof. The first part is obvious since

$$\Psi(\lambda, \sigma_{\max}) = \Psi(\lambda, \sigma_{\max}) * \mathbb{1} = \Psi(\lambda, \sigma_{\max}) * x_0 = x_{\mathcal{S}} \otimes e^0.$$

For the second part, observe that

$$x_{I^c} \otimes e^{A_I} = \Psi(\lambda, \sigma_{\max}) * x_I = x_{\mathcal{S}} * x_I.$$

□

The next paragraph is dedicated to discussing an example where the symplectic manifold has a semi-free circle action but the Seidel automorphism has higher order terms when evaluated on a particular class. In this example the fixed points are along submanifolds. This illustrates that we cannot have a result similar to Theorem 3.13 if we weaken one of our hypotheses.

3.5. **Example** ([4, Example 5.6]). Let $M = \widetilde{\mathbb{P}^2}$ be the one-point blow up of $\mathbb{P}^2$ with the symplectic form ω_μ so that on the exceptional divisor E, $0 < \omega_\mu(E) = \mu < 1$ and if $L = [\mathbb{P}^1]$ is the standard line, we have $\omega_\mu(L) = 1$. We can identify M with the space

$$\{(z_1, z_2) \in \mathbb{C}^2 | \ \mu \leq |z_1|^2 + |z_2|^2 \leq 1\},$$

where the boundaries are collapsed along the Hopf fibres. One of the collapsed boundaries is identified with the exceptional divisor, the other with L.

A basis for $H_*(M)$ is given by the class of a point pt, the exceptional divisor E, the fibre class $F = L - E$ and the fundamental class $[M]$. Note that the intersection products are given by $E \cdot E = -1$, $E \cdot F = 1$, $F \cdot F = 0$. Denote by b and f the Poincaré duals of E, F respectively. Then $b \cdot b = -1$ and $f \cdot f = 0$. It is not hard to see that the positive generator of $H^4(M)$ is $b \smile f = \mathrm{PD}(pt)$. Let us denote this class by just bf, so that a basis for the cohomology ring is $\{\mathbb{1}, b, f, bf\}$. Observe that M with the usual complex structure is Fano.

The non-vanishing Gromov-Witten invariants are given by

$$\mathrm{GW}^M_{L,3}(bf, bf, f) = \mathrm{GW}^M_{F,3}(bf, b, b) = 1,$$
$$\mathrm{GW}^M_{E,3}(c_1, c_2, c_3) = \pm 1 \text{ where } c_i = b \text{ or } f.$$

Let us consider the usual Novikov ring Λ_ω as the quantum coefficients. Then the quantum products are given by

$$\begin{array}{lllllll} bf * bf &=& (b+f) \otimes e^L, & \quad & bf * f &=& \mathbb{1} \otimes e^L, \\ bf * b &=& f \otimes e^F, & & b * b &=& -bf + b \otimes e^E + \mathbb{1} \otimes e^F, \\ b * f &=& bf - b \otimes e^E, & & f * f &=& b \otimes e^E. \end{array}$$

In [4] it is proved that the circle action on M which is given by

$$\alpha : (z_1, z_2) \mapsto (e^{-2\pi i t} z_1, e^{-2\pi i t} z_2), \text{ for } 0 \leq t \leq 1,$$

is Hamiltonian. The maximum set of this action is exactly the points lying on the exceptional divisor E, and the minimum set is the line L. After taking an appropriate reference section σ, the Seidel element $\Psi(\alpha, \sigma)$ is given by

$$\Psi(\alpha, \sigma) = b.$$

Thus, evaluating the Seidel map on the class f we have

$$\Psi(\alpha, \sigma)(f) = \Psi(\alpha, \sigma) * f = b * f = bf - b \otimes e^E.$$

Therefore the Seidel automorphism does have higher order terms when evaluated on the class f.

Proof of the main result. Now we are ready to prove the main theorem. Recall that the quantum coefficient ring is $\Lambda = \mathbb{Q}[q_1, \ldots, q_n]$. We also denote the usual cup product $a \smile b$ by ab for all $a, b \in H^*(M)$. □

Proof of Theorem 1.1

This is an immediate consequence of the next lemma.

Lemma 4.1. *Let $I = \{1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$, and let $1 \leq i \leq n$. Then*

$$x_{i_1} * \cdots * x_{i_k} = x_I \text{ and } x_i * x_i = \mathbb{1} \otimes e^{A_i} = q_i. \tag{15}$$

Proof. To prove the first equality we will proceed by induction. Assume we have only two elements, say x_i, x_j, with $i \neq j$. Then, by Proposition 3.1 and Remark 3.2 we have

$$x_i * x_j = x_{\{ij\}} + c\ \mathbb{1} \otimes e^B,$$

where the coefficient c is a rational number and $c_1(B) > 0$.

From Corollary 3.14 and the associativity of quantum multiplication we get

$$\begin{aligned}(x_{\mathcal{S}} * x_i) * x_j &= (x_{\{i\}^c} * x_j) \otimes e^{A_i} \\ &= x_{\mathcal{S}} * (x_i * x_j) = x_{\{ij\}^c} \otimes e^{A_{ij}} + c\ x_{\mathcal{S}} \otimes e^B.\end{aligned} \tag{16}$$

By Proposition 3.1 the term $x_{\{i\}^c} * x_j$ is of the form

$$x_{\{i\}^c} x_j + \sum_{c_1(B')>0} a_{B'} \otimes e^{B'},$$

where again $\deg(a_{B'}) = \deg(x_{\{i\}^c}) + \deg(x_j) - 2c_1(B') < 2n$. Since $j \in \{i\}^c$, the term $x_{\{i\}^c} x_j$ is zero. Thus we have

$$\sum_{c_1(B')>0} a_{B'} \otimes e^{B'} \otimes e^{A_i} = x_{\{ij\}^c} \otimes e^{A_{ij}} + c\ x_{\mathcal{S}} \otimes e^B.$$

Then by comparing the degree of the coefficients in the previous equation, the constant c must vanish.

For the general case we will use the same argument. Assume the result holds for k different elements. Let $I' = \{i_{k+1}\} \cup I$. The quantum product $x_{i_1} * \cdots * x_{i_{k+1}}$ is by the inductive hypothesis, the same as $x_I * x_{i_{k+1}}$. This element can be written in terms of the basis as

$$x_I * x_{i_{k+1}} = x_{I'} + \sum_{c_1(B)>0, J \subset \mathcal{S}} a_{B,J}\ x_J \otimes e^B,$$

where $2|J| = \deg(x_J) = \deg(x_{I'}) - 2d \leq \deg(x_{I'}) - 4$ and the coefficients $a_{B,J}$ are rational.

As before, using quantum associativity and Corollary 3.14 we get

$$(17)\qquad (x_{\mathcal{S}} * x_I) * x_{i_{k+1}} = (x_{I^c} * x_{i_{k+1}}) \otimes e^{A_I}$$
$$= x_{\mathcal{S}} * (x_I * x_{i_{k+1}}) = x_{I'^c} \otimes e^{A_{I'}} + \sum_{c_1(B)>0, J\subset \mathcal{S}} a_{B,J}\ x_{J^c} \otimes e^{A_J+B}.$$

Here the degree satisfies

$$(18)\qquad \deg(x_{J^c}) = 2n - \deg(x_{I'}) + 2d \geq 2n - \deg(x_{I'}) + 4 = 2(n - |I| + 1).$$

Now, the center term in (17) is written as

$$(x_{I^c} x_{i_{k+1}} + \sum_{c_1(B')>0, K\subset\mathcal{S}} c_{B',K}\ x_K \otimes e^{B'}) \otimes e^{A_I},$$

where we have

$$(19)\qquad \deg(x_K) \leq \deg(x_{I^c}) + \deg(x_{i_{k+1}}) - 4 = 2(n - |I| - 1).$$

Since $i_{k+1} \in I^c$, $x_{I^c} x_{i_{k+1}} = 0$. Finally we have the identity

$$\sum_{c_1(B')>0, K\subset\mathcal{S}} c_{B',K}\ x_K \otimes e^{B'+A_I} = x_{I'^c} \otimes e^{A_{I'}} + \sum_{c_1(B)>0, J\subset\mathcal{S}} a_{B,J}\ x_{J^c} \otimes e^{A_J+B}.$$

By (18) and (19), the coefficients $a_{B,J}$ are zero. This proves the first part of the lemma.

The second part is analogous; just write

$$x_i * x_i = x_i x_i + c\ \mathbb{1} \otimes e^B = c\ \mathbb{1} \otimes e^B.$$

Then multiplying by $x_{\mathcal{S}}$

$$(x_{\mathcal{S}} * x_i) * x_i = (x_{\{i\}^c} * x_i) \otimes e^{A_i} = c\ x_{\mathcal{S}} \otimes e^B.$$

Since $x_{\{i\}^c} * x_i = x_{\mathcal{S}}$, it follows that $c = 1$ and $e^B = e^{A_i}$. □

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